ISRAEL JOURNAL OF MATHEMATICS, Vol. 56, No. 3, 1986

CONNECTIVITY AND CHROMATIC NUMBER OF INFINITE GRAPHS

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ABSTRACT

Every graph with uncountable chromatic number contains for every finite n an n-connected subgraph with infinite degrees which has uncountable chromatic number.

0. Introduction

The study of the chromatic number of infinite graphs was started by P. Erdős and A. Hajnal in [2]. They showed, using a technique of E. W. Miller [5], that if G is a graph with uncountable chromatic number then it contains a complete bipartite graph on n, $\aleph_1$ points, respectively, for every finite n. Notice that this graph is n-connected. P. Erdős and A. Hajnal also showed that G contains an uncountable chromatic subgraph H, such that every vertex of H has an infinite degree in H. An unsolved problem of [2] generalizes this result: if G has uncountable chromatic number then it contains an ω-connected, uncountable chromatic subgraph. Later the result first mentioned was subsequently extended to the following theorem: every uncountable chromatic graph contains a subgraph isomorphic to the following: the vertices are x_s, y_t where s, t span the set of finite 0–1 sequences, and infinite 0–1 sequences with 0 from some place onward, respectively, x_s is joined to y_t whenever $s \subset t$ (see [4]). Observe that in this (countable) graph every vertex has degree $\aleph_0$.

In the first part of this paper we prove that every uncountable chromatic graph contains an n-connected, uncountable chromatic subgraph for every n (Theorem 1). The proof generalizes the technique of Miller and Erdős–Hajnal on strongly almost-disjoint set-systems for systems with no bound on the sizes of

Received June 3, 1985

the sets. The system is the set consisting of the maximal n-connected (spanned) subgraphs, which are, by assumption, $\aleph_0$-chromatic. We show that they can be well-ordered like $\{A_\alpha : \alpha < \lambda\}$ with the property that $\bigcup\{A_\beta : \beta < \alpha\}$ covers only finitely many points in A_α, and the other points of A_α are joined only to finitely many points in $\bigcup\{A_\beta : \beta < \alpha\}$. Now, as each A_α has countable chromatic number, we can color $\bigcup\{A_\alpha : \alpha < \lambda\}$ with $\aleph_0$ colors by transfinite recursion. By the theorem of Erdős–Hajnal, above, $\bigcup\{A_\alpha : \alpha < \lambda\}$ covers the points of G but a set of countable chromatic number, and the proof is complete.

In the second part of the paper we extend this result to the following: if G has uncountable chromatic number, then it contains an n-connected, uncountable chromatic subgraph H such that every point in H has an infinite degree in H (Theorem 2). The proof of this latter result has the same underlying idea as that of Theorem 1 with the main theorem of [4] instead of the one in [2], but it is much more complicated, uses more set theory, namely the technique elaborated in [4]. This is the reason why we give a separate proof for Theorem 1 as well, it is a bit simpler, and it can be understood knowing virtually nothing of set theory. Notice that Theorem 2 is also an immediate corollary of the above-mentioned conjecture of P. Erdős and A. Hajnal.

The proofs of Theorems 1–2 are given in Sections 1–2 and 3–5, respectively.

1. Preliminaries

Assume that $G = (V, E)$ is a graph. For $T \subseteq V$ denote by $G(T)$ the set of those points $x \in V$, for which x is joined to every point of T.

DEFINITION 1. Denote by $\mathscr{A} = \{A_i : i \in I\}$ the system of those subsets of V which span n-connected subgraphs of G and are *maximal* to this property, i.e. if $X \supsetneq A_i$ then X does not span an n-connected subgraph. For $I' \subseteq I$, put $B(I') = \bigcup\{A_i : i \in I'\}$.

LEMMA 1. (a) *If $i \neq j$, then $A_i \not\subseteq A_j$.*

(b) *Every n-connected $A \subseteq V$ is a subset of some A_i.*

(c) *$V - B(I)$ is $\aleph_0$-chromatic.*

(d) *If $i \neq j$, then $|A_i \cap A_j| \leqq n - 1$.*

PROOF. (a) Trivial.

(b) By Zorn's lemma.

(c) From (b) and the theorem of Erdős and Hajnal mentioned in the introduction.

(d) If $|A_i \cap A_j| \geqq n$, then $A_i \cup A_j$ is n-connected: whenever $X \subseteq A_i \cup A_j$,

$|X| \leqq n-1$, $A_i - X$, $A_j - X$ are connected and they have a common vertex, as $X \not\supseteq A_i \cap A_j$. The n-connectedness of $A_i \cup A_j$ contradicts (a) and (b).

From now on let m denote $3n^2$.

LEMMA 2. *There exists a finite number s such that if $A_1, \ldots, A_m \in \mathscr{A}$ then there are no s members $A'_1, \ldots, A'_s \in \mathscr{A}$ with $|A'_i \cap B| \geqq m$; here $B = \bigcup\{A_i : 1 \leqq i \leqq m\}$.*

PROOF. Assume contradictory. It is enough to find a finite number t with the property that for $A_1, \ldots, A_{3n-1} \in \mathscr{A}$ there are no t sets $A'_1, \ldots, A'_t \in \mathscr{A}$ such that for every A'_j there are *different* vertices $x_1, \ldots, x_{3n-1}$ with $x_i \in A_i \cap A'_j$. Once the claim is proved and we have a family like in the assumption, for every fixed A'_j, as $|A_i \cap A'_j| \leqq n-1$ we can select by induction sets and different points with $x_i \in A_i \cap A'_j$ for a certain subset $T_j \subseteq \{1, 2, \ldots, m\}$ which has at least $3n-1$ elements. If $s > tm^{3n}$, then there are t A'_j for which the T_j are the same.

In order to prove the claim, observe first that as $|A'_{j_0} \cap A'_{j_1}| \leqq n-1$ for $j_0 \neq j_1$, the systems $\{x^0_1, \ldots, x^0_{3n-1}\}$ and $\{x^1_1, \ldots, x^1_{3n-1}\}$ chosen for A'_{j_0}, A'_{j_1}, respectively, differ in at least $2n-1$ places. Again, by Ramsey's theorem, if t is large enough, then there are $2n-1$ A'_j for which these sets pairwise differ in the same $2n-1$ co-ordinates.

In other words, we can find $A_1, \ldots, A_{2n-1}$, $A'_1, \ldots, A'_{2n-1} \in \mathscr{A}$ and $x_{ij} \in A_i \cap A'_j$ such that $x_{ij_0} \neq x_{ij_1}$ and $x_{i_0 j} \neq x_{i_1 j}$. But then $X = \bigcup\{A_i, A'_i : 1 \leqq i \leqq 2n-1\}$ spans an n-connected subgraph: if $|Y| \leqq n-1$, then each of the sets $\{A_i - Y, A'_i - Y : 1 \leqq i \leqq 2n-1\}$ is connected, and, if $i_0 \neq i_1$, there is a j with $x_{i_0 j}, x_{i_1 j} \notin Y$, so $A_{i_0} - Y$, $A_{i_1} - Y$ and $A'_j - Y$ are in the same connected component, which easily gives that $X - Y$ is connected.

LEMMA 3. *If $x \notin A_i$, then x is joined to at most $n-1$ vertices of A_i.*

PROOF. Otherwise $A_i \cup \{x\}$ would be a larger n-connected subset of V.

LEMMA 4. *There is a finite number u such that if $A_1, \ldots, A_m \in \mathscr{A}$, then there are at most u different subsets $T \subseteq A_1 \cup \cdots \cup A_m$ with $|T| = m$ such that there is a point $x \notin A_1 \cup \cdots \cup A_m$ joined to each point in T.*

PROOF. As in Lemma 2, it is enough to show that there is a finite v such that if $A_1, \ldots, A_{2n} \in \mathscr{A}$, there are at most v sequences $y_1, \ldots, y_{2n}$ with $y_i \neq y_j$ $(i \neq j)$, $y_i \in A_i$ and that there exists an $x \notin A_1 \cup \cdots \cup A_{2n}$ joined to $y_1, \ldots, y_{2n}$. Again, if there are "too many" sequences, we can find $x_1, \ldots, x_{2n}$ such that for x_i, x_j the sequences corresponding to them differ on the same non-empty index-set, say on

$\{1,\dots,t\}$. That is to say, we have $y_{ij} \in A_j$ $(1 \leqq j \leqq t,\ 1 \leqq i \leqq 2n)$, $z_j \in A_j$ $(t+1 \leqq j \leqq 2n)$, x_i $(1 \leqq i \leqq 2n)$ such that $x_i \notin A_1 \cup \cdots \cup A_{2n}$, x_i is joined to $\{y_{ij}, z_j : 1 \leqq j \leqq 2n\}$ and $y_{i_0 j} \neq y_{i_1 j}$, $y_{ij_0} \neq y_{ij_1}$, $z_{j_0} \neq z_{j_1}$, $y_{ij_0} \neq z_{j_1}$ for $i_0 \neq i_1$, $j_0 \neq j_1$.

We claim that the set $X = A_1 \cup \cdots \cup A_t \cup \{x_1,\dots,x_{2n}, z_{t+1},\dots,z_{2n}\}$ spans an n-connected subgraph (a contradiction, as $t \geqq 1$). Assume $|Y| \leqq n-1$. All sets $A_1 - Y,\dots,A_t - Y$ are connected. If $x_{i_0} \neq x_{i_1}$ are not in Y, then either there is a z_j $(t < j \leqq 2n)$ not in Y or there is a $j \leqq t$ with $y_{i_0 j}, y_{i_1 j} \notin Y$. (Otherwise Y covers $\{z_{t+1},\dots,z_{2n}\}$ and contains for each $j \leqq t$ either $y_{i_0 j}$ or $y_{i_1 j}$. So Y meets either $\{y_{i_0 1},\dots,y_{i_0 t}, z_{t+1},\dots,z_{2n}\}$ or $\{y_{i_1 1},\dots,y_{i_1 t}, z_{t+1},\dots,z_{2n}\}$ in at least $(2n-t)+t/2 = 2n - t/2 \geqq 2n - n = n$ elements.) So $\{x_1,\dots,x_{2n}\} - Y$ is in the same component also, at least one point of them is joined to $A_1 - Y$ (as $Y \not\supseteq \{y_{11},\dots,y_{n1}\}$) and all are joined to $\{y_{t+1},\dots,y_{2n}\} - Y$, and we are done.

Lemma 5. *If $I' \subseteq I$, then the sets $\{i \in I : |A_i \cap B(I')| \geqq m\}$ and $\{T \subseteq B(I') : |T| = m,\ G(T) \not\subseteq B(I')\}$ both have cardinality at most $|I'| + \omega$.*

Proof. Immediate from Lemmas 2 and 4.

2. Subsets of I

Definition 2. A subset $I' \subseteq I$ is *closed*, if for $i \notin I'$ $A_i \cap B(I')$ is finite and every $x \in A_i - B(I')$ is joined to finitely many points of $B(I')$.

Definition 3. A pair (I', I'') with $I' \subseteq I'' \subseteq I$ is *legal* if I' is closed, and

(a) if $|A_i \cap B(I'' - I')| \geqq m$, then $i \in I''$;

(b) if $T \subseteq B(I'' - I')$, $|T| = m$, $|G(T)| \geqq n$, then there is a (unique) $i \in I''$ with $A_i \supseteq T \cup G(T)$;

(c) if $T \subseteq B(I'' - I')$, $|T| = m$, $|G(T)| < n$, then $G(T) \subseteq B(I'')$.

Lemma 6. (a) *If (I', I'') is legal, then I'' is closed.*

(b) *If (I_0, I_2) is legal, $I_0 \subseteq I_1 \subseteq I_2$, and I_1 is closed, then (I_1, I_2) is legal, as well.*

(c) *Assume that γ is an ordinal, $\{I_\alpha : \alpha \leqq \gamma\}$ is a continuous, increasing sequence of subsets of I, (I_0, I_α) is legal for $\alpha < \gamma$, then (I_0, I_γ) is legal.*

Proof. Straightforward from the definitions.

Lemma 7. *If (I_0, I_2) is legal, $X \subseteq I_2 - I_0$, then there is an $I_1 \subseteq I_2$ with (I_0, I_1) legal, $X \subseteq I_1 - I_0$ and $|I_1 - I_0| \leqq |X| + \omega$.*

Proof. Put $X_0 = X$. If X_t is defined for a $t < \omega$, choose $X_{t+1} \supseteq X_t$, $X_{t+1} \subseteq I_2 - I_0$ so that if $|A_i \cap B(X_t)| \geqq m$ then $i \in I_0 \cup X_{t+1}$, if $T \subseteq B(X_t)$, $|T| = m$, $|G(T)| \geqq n$, then there is an $i \in I_0 \cup X_{t+1}$ with $A_i \supseteq T \cup G(T)$, if $|G(T)| < n$

then $G(T) \subseteq B(I_0 \cup X_{t+1})$. Such an $X_{t+1} \subseteq I_2 - I_0$ can be found by Lemma 5 with $|X_{t+1}| \leqq |X_t| + \omega$, and we can take $I_1 = I_0 \cup \bigcup \{X_t : t < \omega\}$.

DEFINITION 4. If $I' \subseteq I$, a *correct order* on I' is a well-order $<$ with the following property: if $i \in I'$ and $B = B(\{j \in I' : j < i\})$, then $A_i \cap B$ is finite and every $x \in A_i - B$ is joined to finitely many points of B.

LEMMA 8. *If (I', I'') is legal, then there exists a correct order on $I'' - I'$.*

PROOF. By induction on $|I'' - I'|$. If $|I'' - I'| \leqq \omega$, say $I'' - I' = \{i_0, i_1, \ldots\}$ order as $i_0 < i_1 < \cdots$. Now the closedness of I' and Lemmas 1(d) and 3 give the result. If $\kappa = |I'' - I'| > \omega$, by Lemmas 6(c) and 7 and a straightforward construction we can define a continuous, increasing chain $\{I_\alpha : \alpha \leqq \kappa\}$ with $I_0 = I'$, $I_\kappa = I''$, $|I_\alpha - I_0| < \kappa$, (I_0, I_α) legal. As $(I_0, I_{\alpha+1})$ is legal so is $(I_\alpha, I_{\alpha+1})$ (Lemma 6(a), (b)), so by the inductive assumption there is a correct order on $I_{\alpha+1} - I_\alpha$, $<_\alpha$. Now put $i < j$ for $i, j \in I'' - I'$ if there is an $\alpha < \kappa$ with either $i <_\alpha j$ or $i \in I_\alpha$, $j \in I_{\alpha+1} - I_\alpha$.

THEOREM 1. *If a graph has uncountable chromatic number, then it has an n-connected uncountable chromatic (spanned) subgraph.*

PROOF. By Lemma 8, as $(\emptyset, I)$ is legal, there is a correct order on I. If every A_i is countable chromatic, we can, along the correct order on I, recursively extend a good coloring with countably many colors to every $B(I')$ where I' runs through the initial segments of I. This gives a good coloring of $B(I)$ with countably many colors. By Lemma 1(c), $V - B(I)$ is countable chromatic, so V is countable chromatic, as well.

3. More preliminaries

In order to prove Theorem 2, first we change some of the definitions according to the extra condition.

DEFINITION 5. If $G = (V, E)$ is a graph, then $\mathcal{D} = \{D_j : j \in J\}$ is the set of those subsets $D \subseteq V$ which span n-connected subgraphs of G and in these subgraphs every point has an infinite degree, moreover, D is *maximal* to this property. For $J' \subseteq J$, put $E(J') = \bigcup \{D_j : j \in J'\}$. Sometimes, if possible, we fix a subsystem of $\mathcal{D}$ as $\{D_\alpha : \alpha < \beta\}$ instead of $\{D_{j_\alpha} : \alpha < \beta\}$.

LEMMA 9. (a) *If $i \neq j$, then $D_i \not\subseteq D_j$.*

(b) *If $A \subseteq V$ spans an n-connected graph with infinite degrees, then $A \subseteq D_i$ for some $i \in J$.*

(c) $V - E(J)$ *has countable chromatic number.*

(d) *If* $i \neq j$, *then* $|D_i \cap D_j| \leqq n-1$.

PROOF. (a) Trivial.

(b) By Zorn's lemma.

(c) By the result of [4] mentioned in the Introduction, every uncountable chromatic graph contains an n-connected subgraph with infinite degrees: take x_s, y_t for those s, t which start with $n-1$ zeroes.

(d) Like in the proof of Lemma 1(d), if $|D_i \cap D_j| \geqq n$, then $D_i \cup D_j$ is n-connected and has infinite degrees.

LEMMA 10. *There exists a finite s such that if* $D_1, \dots, D_m \in \mathcal{D}$ *then there are no* s *members* $D'_1, \dots, D'_s \in \mathcal{D}$ *with* $|D'_i \cap E| \geqq m$ *where* $E = \bigcup\{D_i : 1 \leqq i \leqq m\}$.

PROOF. Exactly as in the proof of Lemma 2.

The crux of the proof of Theorem 2 in that one cannot simply adapt Lemma 4; the points $\{x_1, \dots, x_{2n}, z_{t+1}, \dots, z_{2n}\}$ would have finite degrees.

LEMMA 11. *If* $x \notin D_i$, *then* x *is joined to only finitely many points of* D_i.

PROOF. Otherwise $D_i \cup \{x\}$ would be a larger n-connected subset with infinite degrees.

LEMMA 12. *Assume that* $\kappa > \omega$ *is regular,* $A(\alpha)$ *is a set for* $\alpha < \kappa$, $S \subseteq \kappa$ *is stationary,* ω *many ordinals* $\gamma(\alpha, 0) < \gamma(\alpha, 1) < \cdots < \alpha$ *are given for* $\alpha \in S$, $i < \omega$, *a point* $y(\alpha, i) \in A(\gamma(\alpha, i))$ *is given. Then there are stationary many* $\alpha \in S$ *such that for every* $k < \omega$ *there is a stationary set* $T \subseteq S$ *such that if* $\beta \in T$ *and* $i \leqq k$, *then* $\gamma(\alpha, i) = \gamma(\beta, i)$, *and, moreover, either* $y(\alpha, i) = y(\beta, i)$ *for* $\beta \in T$, *or the points* $\{y(\beta, i) : \beta \in T\}$ *are different from each other and from* $y(\alpha, i)$.

PROOF. We can assume that $|A_\alpha| \leqq \kappa$ for $\alpha < \kappa$. For every $\alpha < \kappa$ fix a well-ordering of $A(\alpha)$ of type $\leqq \kappa$; we will only be interested in the index $\delta(\alpha, i)$ of the point $y(\alpha, i)$ in $A(\gamma(\alpha, i))$. It is enough to guarantee a stationary $T \subseteq S$ with $\gamma(\alpha, i) = \gamma(\beta, i)$ for $i \leqq k$, $\beta \in T$ and either $\delta(\alpha, i) < \alpha$ and $\delta(\alpha, i) = \delta(\beta, i)$ or $\delta(\alpha, i) \geqq \alpha$ and $\delta(\beta, i) \geqq \beta$ $(\beta \in T)$.

Assume that this last claim does not hold, i.e. there exists a closed, unbounded $C \subseteq \kappa$, and for every $\alpha \in S \cap C$ there is a closed, unbounded $C_\alpha \subseteq \kappa$ and a $k(\alpha) < \omega$ such that if $\beta \in C_\alpha \cap S$, there is an $i \leqq k(\alpha)$ such that either $\gamma(\alpha, i) \neq \gamma(\beta, i)$ or $\delta(\alpha, i) \neq \delta(\beta, i) < \alpha$ or $\delta(\alpha, i) \geqq \alpha$, but $\delta(\beta, i) < \beta$.

For a stationary $T \subseteq S$, $k(\alpha) = k$. Put $U = T \cap \nabla\{C_\alpha : \alpha \in T\}$. Using the pressing-down lemma, there is a stationary set $V \subseteq U$, and there are ordinals $\gamma_i, \delta_i, \varepsilon_i \in 2$ $(i \leqq k)$, such that for $\alpha \in V$, $i \leqq k$, $\gamma(\alpha, i) = \gamma_i$ holds, and, if $\varepsilon_i = 0$, then $\delta(\alpha, i) = \delta_i$, if $\varepsilon_i = 1$, then $\delta(\alpha, i) \geqq \alpha$. Now, if we choose $\alpha < \beta$ from V, $\beta \in C_\alpha \cap S$, and this contradicts our assumptions.

4. Subsets of J

Definition 6. If $J' \subseteq J'' \subseteq J$, then J' is *relatively closed* in J'', if for $i \in J'' - J'$, $D_i \cap E(J')$ is finite and every $x \in D_i - E(J')$ is joined to only finitely many points of $E(J')$. $J' \subseteq J$ is *closed*, if it is relatively closed in J.

Definition 7. If $J' \subseteq J$, a *correct order* on J' is a well-order $<$ with the following property: if for $i \in J'$, $E = E(\{j \in J' : j < i\})$ then $D_i \cap E$ is finite and every $x \in D_i - E$ is joined to finitely many points of E.

Lemma 13. *Assume that $J' \subseteq J$ is the continuous, increasing union of $\{J_\alpha : \alpha < \gamma\}$ with J_α relatively closed in $J_{\alpha+1}$ $(\alpha < \gamma)$, and every $J_{\alpha+1} - J_\alpha$ has a correct order. Then so has J'.*

Proof. We can fuse the well-orders as in Lemma 8.

Lemma 14. *Assume $J' \subseteq J$, $|J'| = \kappa > \omega$, regular, and if $|D_i \cap E(J')| \geqq m$ then $i \in J'$. Identify J' as κ. Then $N = \{\alpha < \kappa : \alpha$ is not relatively closed in $\kappa\}$ is non-stationary.*

Proof. Assume that N is stationary. Lemma 10 gives a closed, unbounded set $C \subseteq \kappa$ such that if $\gamma \in C$ and $\beta \geqq \gamma$ then $|D_\beta \cap E(\gamma)| < m$. Therefore there is a stationary $S \subseteq N$ such that if $\alpha \in S$ then there exists a $\beta(\alpha) \geqq \alpha$ and an $x(\alpha) \in D_{\beta(\alpha)} - E(\alpha)$ which is joined to infinitely many points in $E(\alpha)$. Fix, for $\alpha \in S$, $\beta(\alpha)$ and $x(\alpha)$; we can even assume that the mapping $\alpha \mapsto \beta(\alpha)$ is one-to-one. Using Lemma 11 there are infinitely many ordinals $\gamma(\alpha, i) < \alpha$ such that there is a point $y(\alpha, i) \in D_{\gamma(\alpha,i)}$ joined to $x(\alpha)$. We can assume that $\gamma(\alpha, i) = \gamma_i$ for $i \leqq m$, $\alpha \in S$; choose $\delta \in C$ with $\delta > \gamma_i$ $(i \leqq m)$. By induction on $t < \omega$ we choose ordinals $\delta < \alpha_0 < \alpha_1 < \cdots < \alpha_t < \cdots$ $(\alpha_t \in S)$ such that the following stipulations are satisfied:

(1) for every $k, t < \omega$ there is a stationary $S_{t,k} \subseteq S$ such that for $\alpha \in S_{t,k}$, $\gamma(\alpha, i) = \gamma(\alpha_t, i)$ $(i \leqq k)$, and, for fixed $i \leqq k$, the points $\{y(\alpha, i) : \alpha \in S_{t,k}\}$ are either different or equal to $y(\alpha_t, i)$;

(2) for every $k, t < \omega$ there is an N with $t < N < \omega$ and $\alpha_N \in S_{t,k}$.

This can easily be done by induction, using Lemma 12. Now put $X = \bigcup\{X_{t,k} : t, k < \omega\} \cup \{x(\alpha_0), x(\alpha_1), \dots\}$ where $X_{t,k}$ is either $D(\gamma(\alpha_t, k))$ or

$\{y(\alpha_t, k)\}$ according to whether in (1) the first or second possibility holds. We claim that X spans an n-connected subgraph with infinite degrees. First we check degrees. $x(\alpha_t)$ is joined to all $y(\alpha_t, k)$ $(k < \omega)$. $y(\alpha_t, k)$ is joined to those $x(\alpha_N)$ for which (1) holds with $k = m+1, m+2, \ldots$, etc. The points in $D(\gamma(\alpha_t, k))$ have infinite degrees, as well.

Assume that $Y \subseteq X$, $|Y| \leqq n-1$. The subsets $X_{t,k} - Y$ are connected. If $x(\alpha_t)$, $x(\alpha_{t'}) \not\in Y$, then they are in different components of $X - Y$ only if $y(\alpha_t, k) \in Y$ for $k \leqq m$ whenever $y(\alpha_t, k) = y(\alpha_{t'}, k)$; either $y(\alpha_t, k) \in Y$ or $y(\alpha_{t'}, k) \in Y$ when they are different. But this is impossible as we have seen in the proof of Lemma 4.

The set $X_{t,k} = \{y(\alpha_t, k)\}$ is joined to infinitely many $x(\alpha_N)$, and only $n-1$ of them can be in Y. If $X_{t,k} = D(\gamma(\alpha_t, k))$, the infinitely many different points $x(\alpha_N)$ are joined to the different $y(\alpha_N, k)$, only $n-1$ can be in Y, so there is an $N < \omega$ such that $x(\alpha_N)$ is joined (in $X - Y$) to $X_{t,k}$, and we are done.

As X is n-connected and has infinite degrees, there is an i with $D_i \supseteq X$. By the assumption in the lemma, $i \in J'$. By the choice of δ, and by $|X \cap E(\delta)| \geqq m$, $i < \delta$ holds, so $X \subseteq E(\delta)$ which is impossible, as none of the points $x(\alpha_t)$ is in $E(\delta)$.

Lemma 15. *If $X \subseteq J_1 \subseteq J$ then there exists J_0, a relatively closed subset of J_1, with $X \subseteq J_0$, $|J_0| \leqq |X| + \omega$.*

Proof. Put $\kappa = |X| + \omega$. By induction on $\alpha < \kappa^+$, we choose the sets X_α with $X \subseteq X_\alpha$, $|X_\alpha| \leqq \kappa$, such that for every α, if $i \in J$ and $|D_i \cap E(X_\alpha)| \geqq m$ then $i \in X_\alpha$, $\{X_\alpha : \alpha < \kappa^+\}$ increasing, continuous, and if X_α is not relatively closed in J_1 (and we can assume this) an $i \in X_{\alpha+1} - X_\alpha$ with a point $x \in D_i - E(X_\alpha)$ is joined to infinitely many points in $E(X_\alpha)$. But this is impossible, by Lemma 14. This gives a closed $J' \supseteq X$ with $|J'| \leqq |X| + \omega$. Now $J_0 = J' \cap J_1$ is relatively closed in J_1.

Definition 8. If $<$ is a correct order on $J' \subseteq J$, then $X \subseteq J'$ is *neat* if the following holds: whenever $i \in X$, $j < i$ and $D_j \cap D_i \neq \emptyset$ or a point in $D_i - E(\{j : j < i\})$ is joined to a point in D_j, then $j \in X$.

Lemma 16. *If X is a neat subset of J', then X is relatively closed in J'.*

Proof. If $i \in J' - X$ and $|D_i \cap E(X)| \geqq \omega$ then, as $<$ is a correct order, there is a $j \in X$ with $i < j$, $D_i \cap D_j \neq \emptyset$. If $i \in J' - X$ and $x \in D_i - E(X)$ is joined to $y_0, y_1, \ldots$ in $E(X)$, for every y_j choose the $<$-minimal t_j for which $y_j \in D_{t_j}$ ($t_j \in X$ as X is neat), by Lemma 11, infinitely many of the t_j are different, so there is a t_j with $i < t_j$, and $x \in D_i$ and $y_j \in D_{t_j} - E(\{r : r < t_j\})$ are joined, which gives $i \in X$.

LEMMA 17. *If $X \subseteq J' \subseteq J$, and J' has $<$, a correct order, then there is a neat subset $Y \subseteq J'$ with $X \subseteq Y$, $|Y| \leqq |X| + \omega$.*

PROOF. From Definitions 7 and 8 and an easy closure argument.

LEMMA 18. *Assume that $\{J_\alpha : \alpha \leqq \gamma\}$ is an increasing, continuous sequence with J_α relatively closed in $J_{\alpha+1}$ ($\alpha < \gamma$), $J_{\alpha+1} - J_\alpha$ has a correct order. Then $J_\gamma - J_0$ has a correct order.*

PROOF. We can fuse these correct orders.

5. The proof of Theorem 2

First we give a proof on singular compactness of the correct order property. This is a special case of Shelah's famous Singular Cardinal Compactness Theorem [6], but it is not easy to see that it is. Even the simpler expositions [1], [3] are complicated enough that to give a proof (extracted from [3]) is simpler than to see that the conditions of the theorem really apply. Anyway, the next lemma should be best attributed to Saharon Shelah.

LEMMA 19. *Assume that $J' \subseteq J$, $|J'| = \lambda > \mathrm{cf}(\lambda)$, J' is closed. If for every $J'' \subseteq J'$, $|J''| < \lambda$, J'' has a correct order, then so has J'.*

PROOF. Let $\{\lambda_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ be an increasing, continuous sequence of cardinals converging to λ, $\lambda_0 > \mathrm{cf}(\lambda)$. Put $J' = \bigcup\{J_\alpha : \alpha < \mathrm{cf}(\lambda)\}$ as increasing decomposition with $|J_\alpha| = \lambda_\alpha$. Our aim is to find another continuous decomposition $\{X^\alpha : \alpha < \mathrm{cf}(\lambda)\}$ with correct orders $<^{\alpha+1}$ on $X^{\alpha+1}$ such that X^α is neat in $X^{\alpha+1}$ by $<^{\alpha+1}$ and therefore Lemmas 16 and 18 give the claim.

Put $X^\alpha_0 = J_\alpha$. If X^α_k is found with $|X^\alpha_k| = \lambda_\alpha$ for a $k < \omega$, choose a closed $Y^\alpha_k \supseteq X^\alpha_k$ with $|Y^\alpha_k| = |X^\alpha_k|$. Let $<^\alpha_k$ be a correct order which end-extends $<^\alpha_{k-1}$ (possible as Y^α_{k-1} is closed). Enumerate Y^α_k as $Y^\alpha_k = \{y^{\alpha,k}_\xi : \xi < \lambda_\alpha\}$. Choose

$$X^\alpha_{k+1} \supseteq Y^\alpha_k \cup \{y^{\beta,k}_\xi : \beta < \mathrm{cf}(\lambda), \xi < \lambda_\alpha\}$$

with the properties $|X^\alpha_{k+1}| = \lambda_\alpha$, $X^\alpha_{k+1} \supseteq \bigcup\{X^\beta_{k+1} : \beta < \alpha\}$ and $X^\alpha_{k+1} \cap Y^{\alpha+1}_l$ is a neat subset of $Y^{\alpha+1}_l$ by $<^{\alpha+1}_l$ for $l \leqq k$ (possible by Lemma 17, and a straightforward ω-long construction). Put $X^\alpha = \bigcup\{X^\alpha_k : k < \omega\} = \bigcup\{Y^\alpha_k : k < \omega\}$. $<^\alpha = \bigcup\{<^\alpha_k : k < \omega\}$ is a correct order on X^α. Clearly, $\{X^\alpha : \alpha < \mathrm{cf}(\lambda)\}$ is an increasing decomposition of J'. It is continuous, for, if $\alpha < \mathrm{cf}(\lambda)$ is limit, and $x \in X^\alpha$, say

$x \in Y^\alpha_k$, $x = y^{\alpha,k}_\xi$ with $\xi < \lambda_\alpha$. As α is limit, there is a $\beta < \alpha$ with $\xi < \lambda_\beta$. Therefore $x \in X^\beta_{k+1} \subseteq X^\beta$.

X^α is neat in $X^{\alpha+1}$ by $<^{\alpha+1}$, as $X^\alpha = X^\alpha \cap X^{\alpha+1} = \bigcup\{X^\alpha \cap Y^{\alpha+1}_l : l < \omega\} = \bigcup\{\bigcup\{X^\alpha_k \cap Y^{\alpha+1}_l : l \leqq k < \omega : l < \omega\}$. If l is fixed, the sets $X^\alpha_k \cap Y^{\alpha+1}_l$ $(k = l, l+1, \dots)$ are neat subsets of $Y^{\alpha+1}_l$, therefore $\bigcup\{X^\alpha_k \cap Y^{\alpha+1}_l : l \leqq k < \omega\} = X^\alpha \cap Y^{\alpha+1}_l$ is neat in $Y^{\alpha+1}_l$, so, as the sets $Y^{\alpha+1}_l$ end-extend each other, X^α is neat in $\bigcup\{Y^{\alpha+1}_l : l < \omega\} = X^{\alpha+1}$.

LEMMA 20. *For every $J' \subseteq J$ there is a correct order on J'.*

PROOF. By induction on $|J'|$. If J' is countable, it is trivial from Lemmas 9(d) and 11. If $|J'|$ is singular, use Lemma 19. If $|J'|$ is regular, use Lemmas 14 and 18.

THEOREM 2. *Every uncountable chromatic graph contains an uncountable chromatic n-connected subgraph with all degrees infinite.*

PROOF. Like in Theorem 1.

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